

# Spectrum and Statistical Entropy of AdS Black Holes

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## ABSTRACT

Popular approaches to quantum gravity describe black hole microstates differently and apply different statistics to count them. Since the relationship between the approaches is not clear, this obscures the role of statistics in calculating the black hole entropy. We address this issue by discussing the entropy of eternal AdS black holes in dimension four and above within the context of a midisuperspace model. We determine the black hole eigenstates and find that they describe the quantization in half integer units of a certain function of the Arnowitt-Deser-Misner (ADM) mass and the cosmological constant. In the limit of a vanishing cosmological constant (the Schwarzschild limit) the quantized function becomes the horizon area and in the limit of a large cosmological constant it approaches the ADM mass of the black holes. We show that in the Schwarzschild limit the area quantization leads to the Bekenstein-Hawking entropy if Boltzmann statistics are employed. In the limit of a large cosmological constant the Bekenstein-Hawking entropy can be recovered only via Bose statistics. The two limits are separated by a first order phase transition, which seems to suggest a shift from “particle-like” degrees of freedom at large cosmological constant to geometric degrees of freedom as the cosmological constant approaches zero.

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## I. INTRODUCTION

According to the results of Bekenstein, Hawking and others [1, 2, 3, 4, 5], a matter cloud of one solar mass would gain entropy by a factor of roughly  $10^{20}$  in collapsing to form a black hole. The degrees of freedom responsible for this dramatic increase in entropy are attributed to quantum gravity. About a decade ago it was believed that a successful microcanonical derivation of the Bekenstein-Hawking entropy of a black hole would strongly support the particular theory of quantum gravity from which the black hole microstates were derived, but subsequent research has shown that various seemingly different approaches to quantum gravity successfully reproduce the black hole entropy from a microcanonical or a canonical ensemble. However there are significant variations in the details of the calculations.

Three of the most popular candidates for black hole microstates are duals of weak field string and D-brane states in string theory [6, 7, 8, 9, 10], states of a boundary conformal field theory (CFT) [11, 12, 13, 14, 15, 16] and punctures of a spin network on the horizon in loop quantum gravity [17, 18, 19, 20]. The microstates being counted differ between approaches and there are also differences in the statistics employed to count them. In the string and AdS/CFT approaches, the microstates are counted using Bose statistics, but in loop quantum gravity they are counted using Boltzmann statistics. In fact, if the loop states are assumed indistinguishable one obtains an entropy that is proportional to the square root of the horizon area [21]. It is quite likely that the ability of different approaches to reproduce the Bekenstein-Hawking entropy is suggestive of a “universality” of black hole entropy as proposed in [22], but it is nevertheless of interest to understand the application of different statistics in the counting of the microstates better. Unfortunately, because the relationships between approaches are poorly understood, it has been difficult to address this issue until now.

Here we discuss the role of statistics in the context of a single model of the black hole, applying the results of a midisuperspace quantization program describing the spherical collapse of inhomogeneous dust in any number of spatial dimensions both with and without a negative cosmological constant [23, 24]. While the model is not derived from any deeper theory of quantum gravity such as string theory or loop quantum gravity, it is reasonable to expect that it can adequately address at least the semi-classical features of black holes. We ask what statistics are required to recover the leading behavior of the black hole entropy.

The black hole eigenstates are described by the quantization in half integer units of a certain function,  $\mathcal{A}(M, l)$ , of its mass and the cosmological constant ( $\Lambda = -l^{-2}$ ). If we set  $x_h = R_h/l$ , where  $R_h$  is the area radius of the horizon, we find that when  $x_h^2 \ll 2n(n+1)$  the quantity  $\mathcal{A}(M, l)$  approaches the horizon area. The Schwarzschild black hole belongs in this region and we shall refer to this as the Schwarzschild limit. Thus it is the area of the Schwarzschild horizon that admits a linear spectrum, in keeping with Bekenstein’s original proposal [1]. On the other hand when  $x_h^2 \gg 2n(n+1)$ , the quantity  $\mathcal{A}(M, l)$  approaches the ADM mass and it is the black hole *mass* that admits the same linear spectrum. We show that, to recover an area dependence coinciding with the Bekenstein-Hawking entropy (to leading order, modulo fluctuations), one must use Boltzmann statistics in the first limit and Bose statistics in the second limit. Further, the Bose partition function is dominated by a

contribution from the Schwarzschild limit, which enters via a well known duality linking its high temperature behavior to its low temperature dynamics [25].

In section II we briefly recall some of the basic results from the canonical formulation of  $d = 2 + n$  dimensional spherically symmetric gravity as in [24]. For details of the classical solutions and their behavior we refer the reader to [23]. In section III we obtain stationary bound states describing (eternal) black holes and then, in Section IV we use the black hole spectrum to determine the statistical entropy in the two limiting cases described in the previous paragraph. We conclude in Section V.

## II. CANONICAL FORMULATION

We have already described the 2+1 dimensional black hole in detail [26], so here we will concentrate on the case  $n \geq 2$ . The models are represented by solutions of Einstein's equations sourced by the stress tensor  $T_{\mu\nu} = \varepsilon U_\mu U_\nu$ , where  $\varepsilon(t, \rho)$  is the dust proper energy. In any number of dimensions and in comoving coordinates, the solutions are characterized by two arbitrary functions of the radial coordinate  $\rho$ , *viz.*, the “mass function”,  $F(\rho)$ , and the “energy function”,  $E(\rho)$ . In comoving coordinates,  $\rho$  serves as a spatial label for collapsing dust shells. The mass function  $F(\rho)$  represents the mass-energy contained within a dust shell labeled by  $\rho$  and the energy function is related to the initial velocity distribution of the shells. The classical solutions are given by

$$ds^2 = d\tau^2 - \frac{\tilde{R}^2}{1 + 2E} d\rho^2 + R^2 d\Omega_n^2 \quad (1)$$

where  $\Omega_n$  is the solid angle for the  $n$ -sphere,  $R(\tau, \rho)$  is the radius of the  $n$ -sphere and we have used the notation  $\tilde{R}(\tau, \rho) = \partial_\rho R(\tau, \rho)$ . Einstein's equations then give

$$\begin{aligned} \varepsilon(\tau, \rho) &= \frac{(n-1)}{8\pi G_d} \frac{\tilde{F}}{R^n \tilde{R}}, \\ R^{*2} &= -\frac{2\Lambda}{n(n+1)} R^2 + \frac{F(\rho)}{R^{n-1}} + E(\rho) \end{aligned} \quad (2)$$

where  $G_d$  is the  $d$ -dimensional gravitational constant,  $\tilde{F} = \partial_\rho F(\rho)$  and  $R^*(\tau, \rho) = \partial_\tau R(\tau, \rho)$ . Shells labeled by  $\rho$  become singular as the area radius approaches zero. These classical solutions have been analyzed in [23].

The general spherically symmetric ADM metric

$$ds^2 = N^2 dt^2 - L^2 (dr + N^r dt)^2 + R^2 d\Omega_n^2 \quad (3)$$

can now be embedded in the spacetime described by (1). This procedure was shown in [24] to give a canonical description of the collapse in terms of a phase space consisting of the dust proper time,  $\tau(r)$ , the area radius,  $R(r)$ , and the mass density function,  $\Gamma(r)$ , which is defined in terms of the mass function via

$$F(r) = \frac{\tilde{G}_d}{n} M_0 + \frac{\tilde{G}_d}{n} \int_0^r \Gamma(r') dr', \quad (4)$$

where  $\tilde{G}_d = 16\pi G_d/\Omega_n$  and  $M_0$  represents the boundary contribution to the hypersurface action at the center. In dimension four and higher the boundary contribution from the origin is generally set to zero because a non-vanishing mass function at the origin would represent a singular initial configuration corresponding to a point mass at the center<sup>3</sup>, so we restrict our attention to  $M_0 = 0$ . As shown in [24], the boundary terms can be absorbed into a single hypersurface action and the effective constraints of the gravity dust system reduce to

$$\begin{aligned}\mathcal{H}_r &= \tau' P_\tau + R' \overline{P}_R - \Gamma P'_\Gamma \approx 0 \\ \mathcal{H}^g &= P_\tau^2 + \mathcal{F} \overline{P}_R^2 - \frac{\Gamma^2}{\mathcal{F}} \approx 0,\end{aligned}\tag{5}$$

where the prime refers to a derivative with respect to the ADM label coordinate  $r$ , and

$$\mathcal{F} = 1 - \frac{F}{R^{n-1}} + \frac{2\Lambda R^2}{n(n+1)}\tag{6}$$

vanishes on the apparent horizon, passing from negative inside to positive outside. The first of (5) represents the diffeomorphism (momentum) constraint and the second is the Hamiltonian constraint. This simplified form of the constraints was obtained after several canonical transformations on the original (ADM) phase space in the spirit of Kuchař [27], squaring the Hamiltonian of the transformed system and imposing the momentum constraint. Because of the absence of any derivative terms in the Hamiltonian constraint, it is much easier to quantize than the original gravity-dust system and we use it as the starting point for our discussion. In the quantum theory, the apparent horizon is treated as a boundary at which both continuity and differentiability of the wave functional which solves the Wheeler-DeWitt equation are required.

### III. QUANTUM STATES

Dirac's quantization procedure may now be applied to turn the classical constraints into operator constraints on wave-functionals. According to it, the momenta are replaced by functional differential operators (we set  $\hbar = 1$ )

$$\hat{P}_X = -i \frac{\delta}{\delta X(r)},\tag{7}$$

and one may write the quantum Hamiltonian constraint as [28]

$$\hat{\mathcal{H}}\Psi[\tau, R, \Gamma] = \left[ \frac{\delta^2}{\delta \tau^2} + \mathcal{F} \frac{\delta^2}{\delta R^2} + A\delta(0) \frac{\delta}{\delta R} + B\delta(0)^2 + \frac{\Gamma^2}{\mathcal{F}} \right] \Psi[\tau, R, \Gamma] = 0,\tag{8}$$

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<sup>3</sup> The situation is different in 2+1-dimensions. A non-vanishing contribution from the origin is essential to allow for an initial velocity profile that vanishes there. This does not lead to singular initial data and the boundary contribution does not have the interpretation of a point mass situated at the center.

where  $A(R, F)$  and  $B(R, F)$  are smooth functions of  $R$  and  $F$  which encapsulate the factor ordering ambiguities. The divergent quantities  $\delta(0)$  and  $\delta(0)^2$  are introduced to indicate that the factor ordering problem can be dealt with only after a suitable regularization procedure has been implemented. We notice that the Hamiltonian constraint contains no functional derivative with respect to the mass density function. In fact the mass density appears merely as a multiplier of the potential term in the Wheeler-DeWitt equation. This indicates that  $\Gamma(r)$ , and hence the initial energy density distribution,  $\varepsilon(0, r)$  may be externally specified. Once specified,  $\Gamma(r)$  determines the quantum theory of a particular classical model.

The quantum momentum constraint, on the other hand,

$$\hat{\mathcal{H}}_r \Psi[\tau, R, \Gamma] = \left[ \tau' \frac{\delta}{\delta \tau} + R' \frac{\delta}{\delta R} - \Gamma \left( \frac{\delta}{\delta \Gamma} \right)' \right] \Psi[\tau, R, \Gamma] = 0, \quad (9)$$

requires no immediate regularization because it involves only first order functional derivatives. To describe a collapsing cloud with a smooth, non-vanishing matter density distribution over some label set,  $r$ , of non-zero measure the Hamiltonian constraint was regularized on a lattice. Accounting for geometric factors and assuming that the wave-functional is factorizable, the continuum limit of the wave-functional can quite generally be taken as

$$\Psi[\tau, R, \Gamma] = \exp \left[ i \int dr \Gamma(r) \mathcal{W}(\tau(r), R(r), F(r)) \right]. \quad (10)$$

It automatically obeys the momentum constraint provided that  $\mathcal{W}(\tau, R, F)$  has no explicit dependence on the label coordinate  $r$ .

We showed in [28] that, for the wave-functionals to be simultaneously factorizable on the lattice and to obey the momentum constraint in the continuum limit (as the lattice spacing is made to approach zero), they must satisfy not one but three equations,

$$\begin{aligned} \left( \frac{\partial W}{\partial \tau} \right)^2 + \mathcal{F} \left( \frac{\partial W}{\partial R} \right)^2 - \frac{4}{\mathcal{F}} &= 0, \\ \left( \frac{\partial^2}{\partial \tau^2} + \mathcal{F} \frac{\partial^2}{\partial R^2} + A \frac{\partial}{\partial R} \right) W &= 0, \\ B &= 0, \end{aligned} \quad (11)$$

the first of which is the Hamilton–Jacobi equation that was used in earlier studies [29] to describe Hawking radiation in the WKB approximation. The function  $B(R, F)$  in (8) is forced to be identically vanishing. The remaining two equations together with hermiticity of the Hamiltonian constraint uniquely fix the measure,  $\mu$ , that defines an inner product on the Hilbert space, determine  $A(R, F)$  in terms of this measure,

$$A = |\mathcal{F}| \partial_R \ln(\mu |\mathcal{F}|), \quad (12)$$

and also determine  $W(\tau, R, F)$ . The solutions were shown to yield Hawking radiation in [24]. Lattice regularization effectively turns the continuum (midi-superspace) problem into a countably infinite set of decoupled mini-superspace problems; the three equations mentioned earlier are required to ensure a sensible, diffeomorphism invariant continuum limit.

Black holes with ADM mass parameter  $M$  are special cases of the solution in (1), obtained when the mass function is constant,  $F = 2G_d M$ , and the energy function is vanishing. This can be shown directly by a coordinate transformation of (1) from the comoving system  $(\tau, \rho)$  to static coordinates  $(T, R)$ , in which the metric has the standard form,

$$ds^2 = \mathcal{F}(R)dT^2 - \mathcal{F}^{-1}(R)dR^2 - R^2 d\Omega_n^2, \quad (13)$$

where  $\mathcal{F}$  is given in (6). To show this, use the solution  $R = R(\tau, \rho)$  of (2) together with the relationship between dust proper time and Killing time

$$\tau = T - \int dR \frac{\sqrt{1 - \mathcal{F}}}{\mathcal{F}}, \quad (14)$$

which was obtained in [24]. We imagine therefore that the black holes are single shells given by the mass function

$$F(r) = 2G_d M \Theta(r) \quad (15)$$

where  $M$  is the mass at label  $r = 0$  and  $\Theta(r)$  is the Heaviside function. The mass density function [see (4)] is therefore

$$\Gamma(r) = \frac{n\Omega_n}{8\pi} M \delta(r) \quad (16)$$

and, because of the  $\delta$ -distributional mass density, the wave-functional in (10) turns into the wave-*function*

$$\Psi[\tau, R, \Gamma] = e^{i \int_0^\infty dr \Gamma(r) \mathcal{W}(\tau(r), R(r), F(r))} = e^{\frac{in\Omega_n}{8\pi} M \mathcal{W}(\tau, R, F)} \quad (17)$$

where  $\tau = \tau(0)$ ,  $R = R(0)$  and  $F = F(0)$ . The Wheeler-DeWitt equation now becomes the Klein-Gordon equation describing the shell. Taking into account the factor ordering ambiguities and absorbing the  $M$  dependent term, which now renormalizes the potential, into the function  $B(R, F)$  we have

$$\left[ \frac{\partial^2}{\partial \tau^2} + \mathcal{F} \frac{\partial^2}{\partial R^2} + A \frac{\partial}{\partial R} + B \right] e^{\frac{in\Omega_n}{8\pi} M \mathcal{W}(\tau, R, F)} = 0, \quad (18)$$

In contrast with the case in which the mass density is a smooth function over some set of non-zero measure, no regularization is necessary here. This means that no further conditions must be met and therefore that the measure as well as the functions  $A(R, F)$  and  $B(R, F)$  will remain undetermined although the function  $A(R, F)$  will continue to be related to the measure according to (12). Thus two conditions are required to proceed with the quantization of the black holes as described above.

The first condition we impose is one on the measure appropriate to the Hilbert space of wave-functions. In [24] we obtained the Hawking evaporation of a collapsing dust cloud surrounding a pre-existing black hole by taking the dust as a small perturbation to the black hole mass function in (15). The calculation proceeded by evaluating the Bogoliubov coefficient in the near horizon limit outside the horizon. It was noted that crucial to obtaining the correct Hawking temperature is the choice of measure appropriate for eternal black holes.

The measure was obtained from the DeWitt supermetric,  $\gamma_{ab}$  on the configuration space  $(\tau, R)$  and can be read directly from the Hamiltonian constraint

$$\gamma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\mathcal{F}} \end{pmatrix}. \quad (19)$$

It gives  $\mu = 1/\sqrt{|\mathcal{F}|}$ , *i.e.*,

$$\langle \Psi_1, \Psi_2 \rangle = \int \frac{dR}{\sqrt{|\mathcal{F}|}} \Psi_1^\dagger \Psi_2 \quad (20)$$

and, via the hermiticity condition (12), the function

$$A(R, F) = |\mathcal{F}| \partial_R \ln(\sqrt{|\mathcal{F}|}) \quad (21)$$

As long as  $\mathcal{F} \neq 0$  the Wheeler DeWitt equation can now be written as

$$\left[ \frac{\partial^2}{\partial \tau^2} \pm \frac{\partial^2}{\partial R_*^2} + B \right] \Psi = 0 \quad (22)$$

where the positive sign in the above equation refers to the exterior, while the negative sign refers to the interior and  $R_*$  is defined by

$$R_* = \pm \int \frac{dR}{\sqrt{|\mathcal{F}|}}. \quad (23)$$

The second condition arises because we are describing a single shell in this simple quantum mechanical model of an eternal black hole and because  $B(R, F)$  represents an interaction of the shell with itself. We simply demand there are no self interactions, *i.e.*, that  $B(R, F) = 0$ . The quantum evolution is then described by the free wave equation in the interior, but by an elliptic equation in the exterior. This signature change has been noted in other models [30, 31] and occurs because of the behavior of  $\mathcal{F}$ , which passes from positive outside the horizon to negative inside. For the black hole, it means that its wave function is supported in its interior. The spectrum will be determined by the proper radius,  $L_h$ , of the horizon,

$$L_h(M, \Lambda) = \int_0^{R_h} \frac{dR}{\sqrt{|\mathcal{F}|}} \quad (24)$$

where  $R_h$  is its area radius. If we extend the coordinate  $R_*$  to range over  $(-\infty, \infty)$ , thereby avoiding any issues related to a boundary at the center [26], this simple model of a quantum black hole effectively describes a dust shell in a “box” of radius  $2L_h(M, l)$ , which itself depends in a complicated way on its total ADM mass and the cosmological constant. The stationary states describe a spectrum of the form (reintroducing Planck’s constant,  $\hbar$ )

$$\mathcal{A} = \frac{n\Omega_n}{4\pi} (2G_d M) L_h = A_{\text{Pl}} \left( j + \frac{1}{2} \right) \quad (25)$$

where  $j$  is a whole number and  $A_{\text{Pl}} = \hbar G_d$  is the Planck area. Note that  $\mathcal{A}$  is not the horizon area although it has area dimension.

It is not possible to give an analytical expression for  $L_h$ , but we will examine two interesting limits. First, defining  $x = R/l$ , we eliminate the mass,  $M$ , in favor of the horizon length,  $x_h = R_h/l$ , by using the fact that  $x_h$  is a solution of  $\mathcal{F} = 0$ ,

$$\frac{2G_d M}{l^{n-1}} = x_h^{n-1} \left[ 1 + \frac{2x_h^2}{n(n+1)} \right], \quad (26)$$

and express the proper length of the horizon as

$$L_h(x_h, l) = l \int_0^{x_h} \frac{dx}{\sqrt{\left(\frac{x_h}{x}\right)^{n-1} \left[ 1 + \frac{2x_h^2}{n(n+1)} \right] - \frac{2x^2}{n(n+1)} - 1}}. \quad (27)$$

Clearly, the dominant contribution to  $L_h$  comes from the near horizon region, so when  $2x_h^2 \ll n(n+1)$  (in the Schwarzschild limit) the integral may be approximated by

$$L_h(M) \approx l \int_0^{x_h} \frac{dx}{\sqrt{\left(\frac{x_h}{x}\right)^{n-1} - 1}} = f(n) \frac{\pi R_h}{n}. \quad (28)$$

where  $R_h = (2G_d M)^{\frac{1}{n-1}}$  is the area radius of the horizon and

$$f(n) = \frac{n\Gamma\left(\frac{1}{n-1} + \frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{n-1}\right)} \quad (29)$$

is approximately one. Thus inserting  $L_h$  into (25) shows that the spectrum can be described, in accordance with Bekenstein's conjecture, as a quantization of the horizon area ( $A = \Omega_n R_h^n$ ) in half integer multiples of the Planck area according to

$$\mathcal{A}_j = f(n) \frac{A_j}{4} = A_{\text{Pl}} \left( j + \frac{1}{2} \right), \quad (30)$$

up to some dimension dependent factors, all of which are contained in the function  $f(n)$ . We will show in the following section that the equispaced area spectrum predicted by our simplified model of a quantum black hole implies that the entropy, which is associated with (the natural logarithm of) the number of microstates compatible with a given macrostate of the black hole, obeys the Bekenstein-Hawking area law provided that the area quanta are assumed distinguishable. The entropy therefore also admits a discrete and evenly spaced spectrum.

On the other hand in the opposite limit, for which  $2x_h^2 \gg n(n+1)$ ,

$$L_h(l) \approx l \int_0^{x_h} \frac{dx}{\sqrt{\frac{2}{n(n+1)} \left( \frac{x_h^{n+1}}{x^{n-1}} - x^2 \right)}} = \frac{l\pi}{2} \sqrt{\frac{2n}{(n+1)}}. \quad (31)$$

Now inserting  $L_h$  into (25) shows that this model predicts approximate *mass* quantization according to<sup>4</sup>

$$M_j = \frac{4h}{nl\Omega_n} \sqrt{\frac{n+1}{2n}} \left( j + \frac{1}{2} \right). \quad (32)$$

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<sup>4</sup> Mass quantization and not area quantization occurs for the BTZ black hole for all  $l$  [26]. There are no black holes in 2+1 dimensions without a cosmological constant.



We note that the gravitational constant,  $G_d$ , has completely disappeared from the quantization condition. The area spectrum is no longer equispaced and, assuming that the Bekenstein-Hawking area law still holds, neither is the entropy.<sup>5</sup> We will argue that the Bekenstein-Hawking area law holds, provided that Bose statistics and not Boltzmann statistics are employed.

Both limits are verified explicitly when  $n = 3$ , where an exact solution for  $L_h$  is expressible in the simple form

$$L_h(x_h, l) = \sqrt{\frac{3}{2}} l \tan^{-1} \sqrt{\frac{2x_h^2}{3} \left(1 + \frac{x_h^2}{3}\right)}. \quad (33)$$

From this point of view the description of the black hole spectrum changes smoothly from area quantization in half integer units in the Schwarzschild limit, when  $2x_h^2 \ll n(n+1)$ , to mass quantization in half integer units when  $2x_h^2 \gg n(n+1)$ .

#### IV. ENTROPY

Given the spectrum of  $\mathcal{A}$  we now ask for the origin of the degeneracy that leads to the black hole entropy. We examine here how the proposal in [26] may be extended to arbitrary dimensions. We treat the black hole as a single shell with the spectrum in (25). However, this single shell is in fact the end state of many shells that have collapsed to form the black hole. Regardless of their history, we assume that each of the shells then occupies only the levels of (25), contributing some multiple of the Planck area to the total “horizon area”,  $\mathcal{A}$ , of the final state. A black hole microstate is thus a particular distribution of collapsed shells among the available levels. If the distribution of shells is such that  $\mathcal{N}_j$  shells occupy level  $j$ , the black hole’s total “horizon area” becomes

$$\mathcal{A} = A_{\text{Pl}} \sum_j \left(j + \frac{1}{2}\right) \mathcal{N}_j \quad (34)$$

and the (single shell) solution in (13) is to be interpreted as an excitation by  $\mathcal{N} = \sum_j \mathcal{N}_j$  collapsed shells. The number of black hole microstates giving the “area”  $\mathcal{A}$  will depend on assumptions concerning the degeneracy of the microstates.

Since  $\mathcal{A}$  is approximately one quarter the horizon area when  $2x_h^2 \ll n(n+1)$  according to (30), the spectrum in (34) represents the “area ensemble”. Assuming the shells to be distinguishable, the number of states can be written in terms of the total number of “area” quanta,  $\mathcal{Q}$ , and the total number of shells,  $\mathcal{N}$ , as

$$\Omega(\mathcal{Q}, \mathcal{N}) = \frac{(\mathcal{N} + \mathcal{Q} - 1)!}{(\mathcal{N} - 1)! \mathcal{Q}!}, \quad (35)$$

where

$$\mathcal{Q} = \frac{\mathcal{A}}{A_{\text{Pl}}} - \frac{\mathcal{N}}{2} \quad (36)$$

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<sup>5</sup> This statement is also true for the BTZ black hole.

Holding  $\mathcal{A}$  fixed and extremizing the microcanonical entropy,  $S_{\text{micro}} = k_B \ln \Omega$ , with respect to the number of shells gives

$$S_{\text{micro}} = (2k_B f(n) \coth^{-1} \sqrt{5}) \frac{A}{4A_{\text{Pl}}}, \quad (37)$$

which is in excellent (better than 91% for any dimension) agreement with the Bekenstein-Hawking entropy. In addition to the exponential growth in the number of states, the area quantization in (34) ensures that the entropy is effectively quantized in units of the Planck area, as originally proposed by Bekenstein [3]. Loop Quantum Gravity (LQG) does not predict an evenly spaced area spectrum. However, very interesting recent work has unravelled a rich (and unexpected) band structure in the black hole degeneracy spectrum from LQG, leading to just such an *effective* equispacing of the entropy spectrum [32, 33, 34].

The same result can be obtained in the canonical ensemble from the partition function

$$\tilde{Z}(\tilde{\beta}) = \sum_{\mathcal{N}_1 \dots \mathcal{N}_j \dots} g(\mathcal{N}_1, \dots, \mathcal{N}_j \dots) \exp \left[ -\tilde{\beta} A_{\text{Pl}} \sum_j \left( j + \frac{1}{2} \right) \mathcal{N}_j \right] \quad (38)$$

in the saddle point approximation and subject to the constraint that  $\sum_j \mathcal{N}_j = \mathcal{N}$ . Note that here  $\tilde{\beta}$  is *not* the black hole temperature, being conjugate to  $\mathcal{A}$  and not the energy  $M$  of the black hole. Distinguishability of the shells implies that  $g(\mathcal{N}_1, \dots, \mathcal{N}_j \dots) = \mathcal{N}! / \mathcal{N}_1! \dots \mathcal{N}_j! \dots$  and gives

$$\tilde{Z}(\tilde{\beta}) = 2^{-\mathcal{N}} \sinh^{-\mathcal{N}} \left[ \frac{\tilde{\beta} A_{\text{Pl}}}{2} \right] \quad (39)$$

allowing us to compute the canonical entropy in the usual way as

$$\begin{aligned} S_{\text{can}} &= k_B \left[ \ln \tilde{Z}(\tilde{\beta}) \Big|_{\mathcal{A} = -\partial \ln \tilde{Z} / \partial \tilde{\beta}} + \tilde{\beta} \mathcal{A} \right] \\ &= k_B \left[ -\mathcal{N} \ln 2 - \mathcal{N} \ln \sinh \left[ \coth^{-1} \left( \frac{2\mathcal{A}}{\mathcal{N} A_{\text{Pl}}} \right) \right] + \frac{2\mathcal{A}}{A_{\text{Pl}}} \coth^{-1} \left( \frac{2\mathcal{A}}{\mathcal{N} A_{\text{Pl}}} \right) \right]. \end{aligned} \quad (40)$$

Again extremizing with respect to the total number of shells yields

$$S_{\text{can}} = (2k_B f(n) \coth^{-1} \sqrt{5}) \frac{A}{4A_{\text{Pl}}}, \quad (41)$$

in precise agreement with the microcanonical entropy in (37). It is a simple matter to show that had the shells been assumed indistinguishable then the entropy would depend on the square-root of the area [21]. The fact that the area degrees of freedom must be treated as distinguishable runs contrary to our intuition for elementary particles in quantum field theory and calls into question whether “area” is a fundamental quantity in quantum gravity.

In the opposite limit,  $2x_h^2 \gg n(n+1)$ , it is the mass and not the area that is quantized in half integer units according to (32). We therefore consider the “energy ensemble”, in which the total mass of the black hole may be given as

$$M = \frac{4h}{n l \Omega_n} \sqrt{\frac{n+1}{2n}} \sum_j \left( j + \frac{1}{2} \right) \mathcal{N}_j, \quad (42)$$

The partition function is (now  $\beta$  is the inverse temperature of the black hole)

$$Z(\beta) = \sum_{\{\mathcal{N}_1, \dots, \mathcal{N}_j, \dots\}} g(\mathcal{N}_1, \dots, \mathcal{N}_j, \dots) \exp \left[ -\frac{4h\beta}{nl\Omega_n} \sqrt{\frac{n+1}{2n}} \sum_j \left( j + \frac{1}{2} \right) \mathcal{N}_j \right] \quad (43)$$

and this time the area dependence of the Bekenstein-Hawking entropy cannot be obtained by treating the shells as distinguishable. Instead we show below that to recover the correct area dependence they must be treated as indistinguishable. Taking  $g(\mathcal{N}_1, \dots, \mathcal{N}_j, \dots) = 1$ , we must then evaluate

$$Z(\xi) = \prod_{j=0}^{\infty} [1 - e^{-\xi(2j+1)}]^{-1} \quad (44)$$

with

$$\xi = \frac{2h\beta}{nl\Omega_n} \sqrt{\frac{n+1}{2n}}. \quad (45)$$

Exploiting the well known high/low temperature duality of the Bose partition function [25], we obtained in [26] a relationship which explicitly relates the partition function at small argument to the partition function at large argument,

$$Z(\xi) = \prod_{j=0}^{\infty} [1 - e^{-\xi(2j+1)}] = \frac{1}{\sqrt{2}} e^{\frac{\pi^2}{12\xi} + \frac{\xi}{24}} [Z(2\pi^2/\xi)]^{-1}. \quad (46)$$

We will be interested in determining the partition function when  $\xi \ll 1$ . The value of  $Z(2\pi^2/\xi)$  depends on assumptions concerning “ground state” of the system. If we think of  $Z(2\pi^2/\xi)$  as the partition function in the Schwarzschild limit, for which we derived the entropy in (41), we find

$$Z\left(\frac{2\pi^2}{\xi}\right) \approx \exp \left[ -\frac{f(n) \coth^{-1} \sqrt{5} \Omega_n}{2(n-1)hG} \left( \frac{\Omega_n(n-1)\pi^2 l}{f(n) \coth^{-1} \sqrt{5}} \sqrt{\frac{2n}{n+1}} \right)^n \xi^{-n} \right] \stackrel{\text{def}}{=} e^{-\sigma_n \xi^{-n}} \quad (47)$$

and from (46) it follows that

$$\ln Z(\xi) = -\frac{1}{2} \ln 2 + \frac{\pi^2}{12\xi} + \frac{\xi}{24} + \frac{\sigma_n}{\xi^n}, \quad (48)$$

For small values of  $\xi$  and  $n \geq 2$  the right hand side is dominated by the last term, therefore

$$M = -\frac{\partial \ln Z}{\partial \beta} \approx n\sigma_n \left( \frac{2h}{nl\Omega_n} \sqrt{\frac{n+1}{2n}} \right)^{-n} \beta^{-n-1} \quad (49)$$

and we obtain the canonical entropy in the saddle point approximation as

$$S_{\text{can}} = (\ln Z + \beta M)_{M=-\partial \ln Z / \partial \beta} = (8\pi^2)^{\frac{n}{n+1}} \left( \frac{(n-1)}{2(n+1)f(n) \coth^{-1} \sqrt{5}} \right)^{\frac{n-1}{n+1}} \frac{A}{4A_{\text{Pl}}} \quad (50)$$

where, in the limit of large black holes,

$$A = \Omega_n [n(n+1)l^2(GM)]^{\frac{n}{n+1}} \quad (51)$$

is just the area of the horizon. The fact that we have recovered an area law is critical for consistency because we are quantizing Einstein’s gravity, whose equivalence with the Bekenstein-Hawking area law was established in [36]. It justifies our use of the Schwarzschild limit as the “ground state” of the system. We may now examine  $\xi$  more carefully by expressing the inverse temperature  $\beta$  in terms of the horizon surface gravity,

$$\frac{\beta}{l} = \frac{2\pi}{l\kappa} = 4\pi \left[ \frac{n-1}{x_h} + \frac{2x_h}{n} \right]^{-1}. \quad (52)$$

Viewed as a function of  $x_h$ , the right hand side has an absolute maximum value at  $2x_{h,\min}^2 = n(n-1)$ . This implies that  $\xi$  is bounded from above and taking  $\xi \ll 1$  is equivalent to the condition  $l\kappa \gg 1$ , which clearly holds in the limit we are considering.

## V. DISCUSSION

In this paper we have examined two limits of a single model of quantum AdS black holes. There are factor ordering ambiguities in our description, but ambiguities are present in all approaches to date. Choosing factor orderings different from the one we have used would lead to different spectra and so alternate descriptions of the black hole states. We do not know how or even if the descriptions would be connected. The factor ordering employed in this paper is consistent with our description of Hawking radiation in [24]. It does not arise for genuine collapse scenarios, where the matter distribution is taken to be smooth over some interval of the ADM label coordinate,  $r$ . There, the diffeomorphism constraint uniquely fixes the factor ordering as well as the measure on the Hilbert space [35]. It is therefore possible that, once the quantum collapse process leading to the final state is more fully understood, this uniqueness will carry over to a unique description of the end state. This seems to be a worthwhile direction for future research.

We have shown that the canonical theory (with the chosen factor ordering) predicts a linear excitation spectrum of a certain quantity, *i.e.*,  $\mathcal{A}(M, l)$ . However,  $\mathcal{A}(M, l)$  has very different interpretations in the two limits that are also addressed by other approaches to quantum gravity. As the cosmological constant vanishes,  $\mathcal{A}(M, l)$  turns into the horizon area and our model yields equispaced area levels. This spectrum agrees with the predictions of LQG, modulo a further degeneracy due to a quantum number that is responsible for the intrinsic geometry [32, 33, 34], provided the horizon area is large. In the limit of a large cosmological constant,  $\mathcal{A}(M, l)$  is the black hole mass-energy. This spectrum coincides with the string and AdS/CFT approaches. Although the area law was shown long ago to be equivalent to Einstein’s gravity [36] both with and without a cosmological constant, to extract the area dependence of the black hole entropy from the quantization of Einstein’s theory it is necessary to use different statistics in each limit: Boltzmann in the first limit and Bose in the second.

In the Schwarzschild limit, the entropy spectrum of the black hole is equispaced in our model. While this is not naïvely apparent in LQG, recent numerical work has shown the existence of a band structure in the degeneracy that leads to an effective equispaced quantization of the entropy [32, 33, 34]. It would be interesting, but is not clear at this time, if

simplified models such as the one we have worked with in this paper can shed some light on this issue.

One feature of our solutions in the opposite limit is the independence of the spectrum on the gravitational constant,  $G_d$ , in all dimensions. According to (45) and (46), the gravitational constant enters in the thermodynamic description via a duality which connects the partition functions in the two limits. In more than three dimensions, these limits are separated by a first order phase transition that occurs when  $2x_h^2 = n(n-1)$  [37, 38] so they are two different phases of the same thermodynamic system, the former with negative heat capacity and the latter with positive heat capacity.<sup>6</sup> The limit  $2x_h^2 \gg 2n(n+1)$  is more in keeping with quantum field theory both because the heat capacity is positive and because the fundamental “particles” must be assumed indistinguishable. The phase transition may describe a change in the nature of the fundamental degrees of freedom from “field theoretic” to “geometric”.

The states we have obtained are valid for all values of  $x_h$ , yet they are eigenstates of different physical quantities in each limit and must be treated as distinguishable or indistinguishable depending on the physical interpretation of the spectrum. This calls for a better understanding. A microscopic description of the thermodynamics in between these two limits should be illuminating.

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<sup>6</sup> In 2+1 dimensions, the picture of the black hole as a stationary state built out of collapsed shells remains but only mass quantization occurs and one recovers the area law using Bose statistics for all values of the cosmological constant [26]. The heat capacity is positive throughout and no phase transition occurs.

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